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# Fermion Decoupling and the Axial Anomaly on the Lattice

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## Abstract

By an explicit calculation of the continuum limit of the triangle graph amplitude in lattice QED we show that in the axial Ward identity the Adler-Bell-Jackiw (ABJ) anomaly exactly cancels the pseudoscalar density term  $2im\langle\bar{\psi}_x\gamma_5\psi_x\rangle_{a=0}$  in the limit of infinite fermion mass  $m$ . The result, a reflection of decoupling of the heavy fermion, provides a convenient framework for computing the flavor-singlet or U(1) axial anomaly in non-abelian gauge theories on lattice. Our calculations on the lattice are performed using Wilson fermions but the results are general.

**Introduction.** The most well-known model for lattice fermions, the Wilson model [1], solves the problem of species doubling through an *irrelevant term*, the Wilson term, which breaks chiral symmetry explicitly. Explicit breaking of chiral symmetry, turns out to have a more profound reason. According to popular perception [2] it is also necessary to generate, from the lattice regulated model, the ABJ anomaly in perturbation theory in the continuum limit. Indeed, the contribution of the Wilson term to the four-divergence of the axial current is treated as the driving term for the ABJ anomaly [2,3].

To examine the role of the underlying lattice fermion model in generating the ABJ anomaly a convenient and transparent starting point is the condition, in this context, for decoupling of the fermion in the large mass limit from the background gauge field [4],

$$\langle \Delta_\mu J_{\mu 5}(x) \rangle_{a=0} = 2im \langle \bar{\psi}_x \gamma_5 \psi_x \rangle_{a=0} - \lim_{m \rightarrow \infty} \left[ 2im \langle \bar{\psi}_x \gamma_5 \psi_x \rangle_{a=0} \right], \quad (1)$$

where  $a$  is the lattice constant. One recognises Eq.(1) as the Adler condition [5] which states that the triangle graph amplitude should vanish in the limit as the mass of the loop fermion becomes infinite. To establish that the decoupling condition is indeed equivalent to the axial Ward identity one needs the supplementary relation

$$\lim_{m \rightarrow \infty} \left[ 2im \langle \bar{\psi}_x \gamma_5 \psi_x \rangle_{a=0} \right] = \frac{ig^2}{16\pi^2} \epsilon_{\mu\nu\lambda\rho} \text{tr} F_{\mu\nu} F_{\lambda\rho}, \quad (2)$$

where  $F_{\mu\nu}$  is the gauge field tensor. A point to note is that, it is convenient to consider the continuum limit of the pseudoscalar density  $\langle \bar{\psi}_x \gamma_5 \psi_x \rangle_a$  being of dimension three, whereas the contribution of the irrelevant term in the axial Ward identity on lattice is of dimension four.

Our derivation of Eqs.(1) and (2) in lattice QED demonstrates that as long as the underlying lattice fermion model removes doubling completely and is gauge-invariant and local, the ABJ anomaly is generated without reference to the specific form of the irrelevant term. In non-abelian gauge theories on lattice Eq.(1) provides, as we shall see, a simple recipe for deriving the U(1) or flavor-singlet axial anomaly.

**Decoupling in QED.** The key to our analysis is the Rosenberg [6] tensor decomposition of the amplitude of the triangle diagrams (i) and (ii) in continuum QED for axial current  $j_{\lambda 5}(x)$  to emit two photons with momenta and polarisation  $(p, \mu)$  and  $(k, \nu)$ .

$$T_{\lambda\mu\nu}^{(i+ii)} = \epsilon_{\lambda\mu\nu\alpha} k_{\alpha} A(p, k, m) + \epsilon_{\lambda\nu\alpha\beta} p_{\alpha} k_{\beta} [p_{\mu} B(p, k, m) + k_{\mu} C(p, k, m)] + [(k, \nu) \leftrightarrow (p, \mu)]. \quad (3)$$

Gauge invariance relates the Rosenberg form factors A to B and C

$$A(p, k, m) = p^2 B(p, k, m) + p \cdot k C(p, k, m). \quad (4)$$

It is to be noted that the form factors  $B$  and  $C$  are of mass dimension -2, and, therefore, must vanish as  $m^{-2}$  for large fermion mass. Gauge invariance, therefore, guarantees that

$$\lim_{m \rightarrow \infty} (p + k)_{\lambda} T_{\lambda\mu\nu}^{(i+j)} = -\epsilon_{\mu\nu\alpha\beta} p_{\alpha} k_{\beta} \lim_{m \rightarrow \infty} [A(p, k, m) + A(k, p, m)] \quad (5)$$

$$= 0, \quad (6)$$

which is the basis of Eq.(1). In the above, (6) follows from (5) because of (4) and the asymptotic behavior of  $B$  and  $C$ .

On lattice, the decoupling condition (6) should be realised in the continuum limit irrespective of the underlying model for fermion as long as it is free from doublers and local. The form factors  $B$  and  $C$  which are highly convergent amplitudes must coincide with their respective expressions in the continuum in all *legitimate* lattice models. Residual model dependence, if any, can appear only in the form factor  $A$  because of potential logarithmic divergence. This, however, is ruled out by the gauge invariance constraint (4).

In lattice QED with Wilson fermions, the Feynman amplitudes corresponding to the two diagrams (i) and (ii) are :

$$[T_{\lambda\mu\nu}^{(i+ii)}]_a = -g^2 \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4 l}{(2\pi)^4} \text{Tr} \left[ \gamma_{\lambda} \gamma_5 \cos a(l + \frac{k-p}{2})_{\lambda} S(l-p) V_{\mu}(l-p, l) S(l) V_{\nu}(l, l+k) S(l+k) + (k, \nu \leftrightarrow p, \mu) \right], \quad (7)$$

with the fermion propagator  $S(l)$  and the one-photon vertex  $V_{\mu}(p, q)$  given by

$$S(l) = \left[ \sum_{\mu} \gamma_{\mu} \frac{\sin al_{\mu}}{a} + \frac{r}{a} \sum_{\mu} (1 - \cos al_{\mu}) + m \right]^{-1}, \quad (8)$$

$$V_{\mu} = \gamma_{\mu} \cos \frac{a}{2} (p + q)_{\mu} + r \sin \frac{a}{2} (p + q)_{\mu}. \quad (9)$$

where  $r$  is the Wilson parameter.

On lattice there are four additional diagrams with *irrelevant* vertices. As will be evident from the following, they do not contribute in the continuum limit.

The lattice amplitude (7) is superficially linearly divergent. However, the leading term, obtained by setting the external momenta  $p, k = 0$  is odd in the loop momentum  $l$  and, therefore, vanishes due to symmetric integration. The amplitude, therefore, vanishes at least linearly in external momenta as indeed the Rosenberg decomposition suggests and, furthermore, the effective divergence is at most logarithmic.

Our strategy is to consider the derivative of (7) with respect to the fermion mass  $m$  rather than the external momenta  $p, k$  as is common practice

$$[R_{\lambda\mu\nu}^{(i)}]_a \equiv \frac{d}{dm} [T_{\lambda\mu\nu}^{(i)}]_a. \quad (10)$$

Lattice power counting gives a negative integer for the effective degree of divergence of  $[R_{\lambda\mu\nu}^{(i)}]_a$ . One can, therefore, take, thanks to the Reisz theorem [7], the continuum limit of the integrands and evaluate the loop integrals in the entire phase space  $-\infty \leq l_{\mu} \leq \infty$  as in the continuum Feynman amplitudes. In the continuum limit, amplitudes of only two diagrams (i) and (ii) survive and amplitudes with *irrelevant* vertices vanish. The amplitudes  $[R_{\lambda\mu\nu}^{(i)}]_{a=0}$  and  $[R_{\lambda\mu\nu}^{(ii)}]_{a=0}$  are individually Bose-symmetric and hence gauge-invariant:

$$R_{\lambda\mu\nu}^{(i+ii)} = 2 \ R_{\lambda\mu\nu}^{(i)} = 2 \ R_{\lambda\mu\nu}^{(ii)}. \quad (11)$$

which is an unexpected bonus.

The Rosenberg tensor decomposition for  $[R_{\lambda\mu\nu}^{(i+ii)}]_{a=0}$  is given by

$$\begin{aligned} [R_{\lambda\mu\nu}^{(i+ii)}]_{a=0} = 4g^2m \int_{-\infty}^{\infty} \frac{d^4l}{(2\pi)^4} & \left[ \text{Tr}(\gamma_5 \gamma_{\lambda} \gamma_{\mu} \gamma_{\nu} \not{l}) \left( \frac{1}{D} (1 + \frac{k^2}{d_3}) - \frac{1}{d_1 d_3^2} \right) \right. \\ & \left. + \text{Tr}(\gamma_5 \gamma_{\lambda} \gamma_{\nu} \not{l} \not{k}) \frac{2(l_{\mu} - p_{\mu})}{D d_1} + (p, \mu \leftrightarrow k, \nu) \right] \end{aligned} \quad (12)$$

where

$$D = d_1 d_2 d_3 \quad \text{and} \quad d_1 \equiv (l - p)^2 + m^2, \quad d_2 \equiv l^2 + m^2, \quad d_3 \equiv (l + k)^2 + m^2. \quad (13)$$

The four-divergence of the amplitude for the axial vector current is to be obtained from

$$\begin{aligned} [(p + k)_\lambda R_{\lambda\mu\nu}^{(i+ii)}]_{a=0} &= \frac{d}{dm} [(p + k)_\lambda T_{\lambda\mu\nu}^{(i+ii)}]_{a=0} \\ &= -\frac{1}{\pi^2} \epsilon_{\mu\nu\alpha\beta} p_\alpha k_\beta \frac{d}{dm} \int_{0 \leq s+t \leq 1} \frac{m^2}{c^2 + m^2} ds dt, \end{aligned} \quad (14)$$

with

$$c^2 \equiv s(1 - s)p^2 + t(1 - t)k^2 + 2st p.k. \quad (15)$$

The Adler condition (1) determines the *constant of integration*

$$[(p + k)_\lambda T_{\lambda\mu\nu}^{(i+ii)}]_{a=0} = -\frac{1}{\pi^2} \epsilon_{\mu\nu\alpha\beta} p_\alpha k_\beta \left[ \int \frac{m^2}{c^2 + m^2} ds dt - \frac{1}{2} \right]. \quad (16)$$

The ABJ anomaly is identified as the  $m = 0$  limit of the right hand side of (16):

$$\text{ABJ anomaly} = \frac{1}{2\pi^2} \epsilon_{\mu\nu\alpha\beta} p_\alpha k_\beta \quad (17)$$

**U(1) axial anomaly in non-abelian gauge theories.** The representation motivated by the decoupling condition (1)

$$\lim_{m \rightarrow \infty} [2im \langle \bar{\psi}_x \gamma_5 \psi_x \rangle_{a=0}] = \lim_{m \rightarrow \infty} [2im \langle x | \text{Tr} \gamma_5 (\not{D} + W + m)^{-1} | x \rangle_{a=0}] \quad (18)$$

constitutes the starting point of our calculation of the axial anomaly in non-Abelian theories, *e.g.*, lattice QCD. The Dirac operator  $\not{D}$  and the Wilson term  $W$  are given by

$$\begin{aligned} D_\lambda &\equiv \frac{1}{2ia} (e^{ip_\lambda a} U_\lambda - U_\lambda^\dagger e^{-ip_\lambda a}) \\ W &\equiv \frac{r}{2a} \sum_\lambda (2 - e^{ip_\lambda a} U_\lambda - U_\lambda^\dagger e^{-ip_\lambda a}) \end{aligned} \quad (19)$$

where  $U_\lambda \equiv \exp(iagA_\lambda)$  is the link variable with  $A_\lambda \equiv t^a A_\lambda^a$  the gauge potential and  $t^a$  the generators of  $SU(N)$ .

One recognises (18) as the analog of the contribution of the Pauli-Villars fermion to the axial anomaly in continuum  $SU(N)$  theory [8]. As in the continuum, our strategy is to develop the Green function for lattice fermion in a perturbative series:

$$\begin{aligned}
(\not{D} + W + m)^{-1} &= (-\not{D} + W + m)G \\
\text{with } G &= \left( -\not{D}^2 + (W + m)^2 + [\not{D}, W] \right)^{-1} \\
&= G_0 - gG_0VG_0 + g^2G_0VG_0VG_0 + \dots
\end{aligned} \tag{20}$$

where the *free* part

$$G_0 = \left( \sum D_\mu^2 + (W + m)^2 \right)^{-1} = \left[ \sum \frac{\sin^2 ap_\mu}{a^2} + \left( \frac{r}{a} \sum_\mu (1 - \cos ap_\mu) + m \right)^2 \right]^{-1} \tag{21}$$

is of Reisz degree  $-2$  and has the expected continuum limit

$$[G_0]_{a=0} = (p^2 + m^2)^{-1} \tag{22}$$

The potential  $gV$  has three pieces

$$gV = gV_0 + gV_1 + gV_2 \tag{23}$$

of which the first piece  $gV_0$  is independent of  $\gamma$ -matrices, has Reisz degree  $+1$  and non-vanishing continuum limit. The pieces  $gV_1$  and  $gV_2$  contain  $\gamma$ -matrices and each has Reisz degree zero. The continuum limit of  $gV_1$  vanishes

$$(gV_1)_{a=0} = [\not{D}, W]_{a=0} = 0, \tag{24}$$

whereas,

$$(gV_2)_{a=0} = \frac{i}{2} \sigma_{\mu\nu} [D_\mu, D_\nu]_{a=0} = -\frac{i}{2} \sigma_{\mu\nu} F_{\mu\nu} \tag{25}$$

where  $F_{\mu\nu}$  is the field tensor in the continuum.

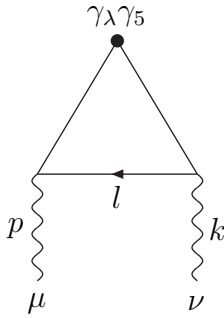
It is clear that the first two terms of the perturbative series (20) do not contribute to the axial anomaly (18) simply because they do not have enough  $\gamma$ -matrices to give non-vanishing Dirac trace. Reisz power counting for the lattice Feynman amplitudes corresponding to

the second and higher order terms in (20) all give negative integers. One can now use the Reisz theorem and take the continuum limit of the integrands in the lattice Feynman amplitudes corresponding to all these terms. Since  $gV_1$  vanishes in the continuum limit it cannot contribute to the axial anomaly. The term which survive in the large mass limit in the continuum is thus given by

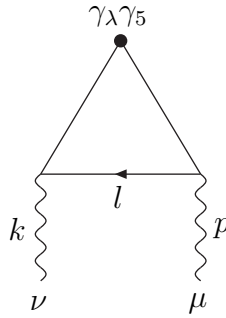
$$\begin{aligned} \lim_{m \rightarrow \infty} [2im \langle \bar{\psi}_x \gamma_5 \psi_x \rangle_{a=0}] &= \lim_{m \rightarrow \infty} [2img^2 \langle x | \text{Tr} \gamma_5 G_0 V_2 G_0 V_2 G_0 | x \rangle_{a=0}] \\ &= \frac{ig^2}{16\pi^2} \epsilon_{\lambda\rho\mu\nu} \text{tr} F_{\lambda\rho}(x) F_{\mu\nu}(x) \end{aligned} \quad (26)$$

where ‘tr’ now denotes trace over internal symmetry indices. Note that the final result (26) is local, all nonlocalities disappearing in the large  $m$  limit, as do all higher order terms in the perturbative series (20).

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(i)



(ii)

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